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Technical Notes

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Vorticity in Unsteady, Viscous, Reacting Flow and Downstream of a Curved Shock

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I. Introduction

UNSTEADY, 3-D, compressible, viscous flows are generally rotational. They may also be diffusive and chemically reacting. Equations for the substantial derivative of the vorticity and for Crocco's equation (often referred to as Crocco's theorem) are generalized. The resulting new relations hold for an unsteady, 3-D flow with body and viscous forces, heat transfer, combustion, diffusion, shock waves, and turbulence. A single-phase, continuum flow is the only significant requirement. The relationship between these two vorticity equations is discussed. Under more restrictive conditions, two new equations are also obtained for the vorticity just downstream of a curved shock wave. The results of the analysis should prove helpful in understanding the role of vorticity in complex flows, whether experimental or computational.

II. Analysis

A. Vorticity Just Downstream of a Shock Wave

Upstream of a curved shock the flow of a perfect gas is assumed to be uniform. A flow plane [1] is defined at a point on the shock by the upstream velocity $\boldsymbol{w}_\infty$ and a downstream pointing unit normal vector $\boldsymbol{n}$. In this plane, β is the shock wave angle (Fig. 1), whose rate of change is

$$\beta' = \frac{d\beta}{d\tilde{s}} \tag{1}$$

where $\tilde{s}$ is arc length along the shock. For a typical detached shock, β decreases with $\tilde{s}$ and β' is negative. If the shape of a 2-D or axisymmetric shock is written as

$$y = f(x) \tag{2}$$

where x is parallel and y is perpendicular to $\boldsymbol{w}_\infty$, then

$$\beta = \tan^{-1}\left(\frac{df}{dx}\right) \tag{3a}$$

$$\beta' = \frac{d^2 f/dx^2}{[1 + (df/dx)^2]^{3/2}} \tag{3b}$$

The curvature of the shock in the flow plane is $-\beta'$.

The vorticity just downstream of a shock, for example, can be written as [2]

$$\omega = w_\infty \frac{\rho}{\rho_\infty}\left(1 - \frac{\rho_\infty}{\rho}\right)^2 \beta' \cos\beta \tag{4}$$

where ρ is the density, also just downstream of the shock. Two new convenient ways of expressing ω are now obtained. First, replace ρ/ρ_∞ by its well-known oblique shock jump equation, to obtain

$$\omega = \frac{2}{\gamma + 1} w_\infty \frac{(M_\infty^2 \sin^2\beta - 1)^2}{M_\infty^2 \sin^2\beta\{1 + [(\gamma - 1)/2] M_\infty^2 \sin^2\beta\}} \beta' \cos\beta \tag{5}$$

where γ is the ratio of specific heats and M_∞ is the freestream Mach number. This relation holds for a 2-D or axisymmetric shock. It is the vorticity counterpart of the usual oblique shock jump conditions; β' is the one new parameter. (The upstream flow is irrotational.) It demonstrates that the vorticity is zero when one of the following conditions holds:

$$\beta = 90 \text{ deg} \tag{6a}$$

$$M_\infty \sin\beta = 1 \tag{6b}$$

$$\beta' = 0 \tag{6c}$$

Condition (6a) occurs at the nose of a detached shock wave. Condition (6b) occurs far downstream where the shock becomes a Mach wave. Condition (6c) occurs when a shock, or a part of a shock, is planar or conical.

The vorticity has an extremum when

$$\frac{\partial\omega}{\partial\tilde{s}} = 0 \tag{7}$$

In view of Eqs. (6) and the negativity of β', the extremum is a negative minimum ω value. Logarithmic differentiation of Eq. (5) yields

$$\frac{1}{\omega}\frac{\partial\omega}{\partial\tilde{s}} = \frac{2(1 + \gamma M_\infty^2 \sin^2\beta)}{(M_\infty^2 \sin^2\beta - 1)\{1 + [(\gamma - 1)/2] M_\infty^2 \sin^2\beta\}} \frac{\beta'}{\tan\beta} - \beta' \tan\beta + \frac{\beta''}{\beta'} \tag{8}$$

Condition (7) provides the location of the extremum as

$$\frac{\beta''}{(\beta')^2} = \tan\beta - \frac{2(1 + \gamma M_\infty^2 \sin^2\beta)}{\tan\beta(M_\infty^2 \sin^2\beta - 1)\{1 + [(\gamma - 1)/2] M_\infty^2 \sin^2\beta\}} \tag{9}$$

where the right side only depends on γ, M_∞, and β.

The vorticity is now determined in terms of β and θ, where θ is the angle of the downstream velocity, relative to $\boldsymbol{w}_\infty$, in the flow plane. It

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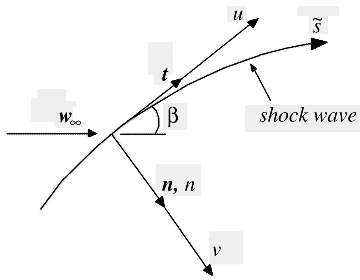

Fig. 1 Section through a 2-D or axisymmetric shock that contains $\boldsymbol{w}_\infty$, $\boldsymbol{n}$, and $\boldsymbol{t}$. The binormal $\boldsymbol{b}$ vector is normal to the page and points outward from the page.

is given by the well-known equation

$$\tan\theta = \cot\beta \frac{M_\infty^2 \sin^2\beta - 1}{1 + [(\gamma+1)/2 - \sin^2\beta] M_\infty^2} \tag{10}$$

Solve this equation for M_∞^2, with the result

$$M_\infty^2 = \frac{1 + \tan\beta\tan\theta}{(1+\tan\beta\tan\theta)\sin^2\beta - [(\gamma+1)/2]\tan\beta\tan\theta} \tag{11}$$

The desired vorticity equation is obtained:

$$\omega = w_\infty \frac{\tan^2\theta}{\sin\beta\cos^2\beta(1+\tan\beta\tan\theta)(\tan\beta - \tan\theta)}\beta' \tag{12}$$

when M_∞^2 is eliminated from Eq. (5).

It is useful to note that $\boldsymbol{\omega}$ is tangent to the shock [3] and parallel to the binormal vector shown in Fig. 1, in which $\boldsymbol{n}$, $\boldsymbol{t}$, and $\boldsymbol{b}$ is a right-handed, orthonormal, shock-fixed basis.

B. Vorticity of a Fluid Particle

As usual, the vorticity and acceleration are

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{w} \tag{13}$$

$$\boldsymbol{a} = \frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t} = \frac{\partial \boldsymbol{w}}{\partial t} + \frac{1}{2}\nabla w^2 + \boldsymbol{\omega} \times \boldsymbol{w} \tag{14}$$

The curl of the acceleration then yields the kinematic relation for the substantial derivative of the vorticity:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \nabla \times \boldsymbol{a} + \boldsymbol{\omega}\cdot(\nabla\boldsymbol{w}) - (\nabla\cdot\boldsymbol{w})\boldsymbol{\omega} \tag{15}$$

The acceleration is provided by the momentum equation

$$\boldsymbol{a} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot\boldsymbol{\tau} + \boldsymbol{F}_b \tag{16}$$

where the shear and body forces are given by

$$\boldsymbol{\tau} = 2\mu\boldsymbol{\varepsilon} + \lambda(\nabla\cdot\boldsymbol{w})\boldsymbol{I} \tag{17a}$$

$$\nabla\cdot\boldsymbol{\tau} = \boldsymbol{F}_s + \nabla[\lambda(\nabla\cdot\boldsymbol{w})] \tag{17b}$$

$$\boldsymbol{F}_s = 2\nabla\cdot(\mu\boldsymbol{\varepsilon}) \tag{17c}$$

$$\boldsymbol{F}_b = -\nabla\Omega \tag{17d}$$

In the preceding equations, $\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}$, μ, λ, $\boldsymbol{I}$, and Ω are the viscous stress tensor, rate-of-deformation tensor, first viscosity coefficient, second viscosity coefficient, unit dyadic, and body-force potential, respectively. When the curl of the acceleration is substituted into Eq. (15), the final result is obtained:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \frac{1}{\rho^2}\nabla\rho\times\nabla p + \boldsymbol{\omega}\cdot(\nabla\boldsymbol{w}) - (\nabla\cdot\boldsymbol{w})\boldsymbol{\omega} - \frac{1}{\rho^2}\nabla\rho\times\{\boldsymbol{F}_s + \nabla[\lambda(\nabla\cdot\boldsymbol{w})]\} + \frac{1}{\rho}\nabla\times\boldsymbol{F}_s \tag{18}$$

Although there are no terms for a body force, diffusion, or chemical reactions, this equation nevertheless covers these cases. The viscous terms are on the right, while $\nabla\rho\times\nabla p$ is zero in a barotropic flow. The equation greatly simplifies if the flow is incompressible. If it is 2-D or axisymmetric, the $\boldsymbol{\omega}\cdot\nabla\boldsymbol{w}$ term is zero.

C. Crocco's Equation

Crocco introduced his vorticity equation [4] for a steady, inviscid, no body force, uniform upstream flow of a simple thermodynamic fluid. It is obtained by eliminating the pressure gradient in the momentum equation by using a $T\,\mathrm{d}s$ thermodynamic equation [5], where T is the temperature and s is the specific entropy. An unsteady form [6] and one for a reacting flow [7] have been derived. Wu and Hayes [7], however, assume a steady flow, thereby excluding the one explicit unsteady term, $\partial\boldsymbol{w}/\partial t$, from Crocco's equation.

For a diffusive and chemically reacting fluid, the appropriate $T\,\mathrm{d}s$ equation is

$$\mathrm{d}h = T\,\mathrm{d}s + \frac{\mathrm{d}p}{\rho} + \sum_i \bar{\mu}_i\,\mathrm{d}n_i \tag{19}$$

Here, h, p, $\bar{\mu}_i$, and n_i are the specific enthalpy, pressure, chemical potential of species i per unit mole, and the mole-mass ratio of species i (i.e., moles of species i per unit mass of mixture), respectively. To derive Crocco's equation, the derivatives in Eq. (19) are replaced with the gradient operator. It might be argued that the derivative symbol should be replaced with a substantial derivative, which follows a fluid particle, that is, a closed thermodynamic system. If this is done, Crocco's equation, with a $\boldsymbol{\omega}\times\boldsymbol{w}$ term, cannot be obtained. Multiplication of the momentum equation by $\boldsymbol{w}\cdot$ eliminates this term. This dilemma is resolved by noting that a $T\,\mathrm{d}s$ equation is convenient but not essential in the derivation.

Virtually all reacting flows of aerodynamic interest are covered by the general ideal gas mixture model [5], whose equations are summarized as follows:

$$p = \frac{\rho\bar{R}T}{W} \tag{20a}$$

$$W = \frac{1}{\sum_i n_i} \tag{20b}$$

$$h = \sum_i n_i \bar{h}_i^0(T) \tag{20c}$$

$$\frac{p_i}{p} = W n_i \tag{20d}$$

$$\bar{s}_i(T, p_i) = \bar{s}_i^0(T, p_{\mathrm{ref}}) - \bar{R}\,\ell n\left(\frac{p_i}{p_{\mathrm{ref}}}\right) \tag{20e}$$

$$s = \sum_i n_i \bar{s}_i(T, p_i) \tag{20f}$$

$$\bar{\mu}_i = \bar{h}_i^0(T) - T\bar{s}_i(T, p_i) \tag{20g}$$

In the preceding equations, $\bar{R}$, W, $\bar{h}_i^0$, p_i, p_{ref}, and $\bar{s}_i^0$ are the universal gas constant, mixture molecular weight, molar enthalpy of species i, partial pressure of species i, reference pressure, and the molar entropy of species i at p_{ref}, respectively. Thermodynamic consistency requires

$$\frac{\mathrm{d}\bar{h}_i^0}{\mathrm{d}T} = T\frac{\mathrm{d}\bar{s}_i^0}{\mathrm{d}T} \tag{20h}$$

One can show that Eqs. (20) identically satisfy Eq. (19). Because Eqs. (20) are algebraic, they can be used with the gradient operator to provide ∇p in the required form for Crocco's equation. With Eqs. (20) equivalent to the $T\,\mathrm{d}s$ equation, it can be used with the thermodynamic derivative replaced by a gradient operator. In conjunction with earlier equations, the general form for Crocco's equation is now readily obtained as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{w} = T\nabla s - \nabla(h_o + \Omega) + \frac{1}{\rho}\boldsymbol{F}_s + \frac{1}{\rho}\nabla[\lambda(\nabla \cdot \boldsymbol{w})] + \sum_i \bar{\mu}_i \nabla n_i \tag{21}$$

where h_o is the stagnation enthalpy and $\boldsymbol{F}_s$ is given by Eq. (17c).

III. Conclusions

Equation (18) can be used to derive an equation for the fluctuating vorticity component in a turbulent flow. For simplicity, an incompressible flow with a constant value for μ is assumed. Equation (18) reduces to

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = \boldsymbol{\omega} \cdot (\nabla \boldsymbol{w}) + \frac{\mu}{\rho}\nabla^2 \boldsymbol{\omega} \tag{22}$$

In a turbulent flow, where

$$\boldsymbol{w} = \boldsymbol{w}_o + \boldsymbol{w}', \qquad \boldsymbol{\omega} = \boldsymbol{\omega}_o + \boldsymbol{\omega}' \tag{23}$$

the fluctuating vorticity component is given by

$$\boldsymbol{\omega}' = \nabla \times \boldsymbol{w}' \tag{24}$$

and

$$\frac{\mathrm{D}\boldsymbol{\omega}'}{\mathrm{D}t} = \boldsymbol{\omega}_o \cdot \nabla \boldsymbol{w}' + \boldsymbol{\omega}' \cdot \nabla \boldsymbol{w}_o + \frac{\mu}{\rho}\nabla^2 \boldsymbol{\omega}' \tag{25}$$

This equation highlights the convective and viscous contributions to changes in the fluctuating component of the vorticity of a fluid particle.

Equations (18) and (21) appear to be disparate. For instance, Eq. (21), but not Eq. (18), contains the entropy, stagnation enthalpy, Ω, and chemical potential terms. Effects due to these parameters, however, can alter Eq. (18), for example, through the $\nabla\rho \times \nabla p$ term. It would appear that Eq. (18) is sufficient to determine the vorticity throughout the flowfield if upstream values are known. This, however, may not be the case. For instance, with a uniform upstream flow, Eq. (18), by itself, is insufficient for determining $\boldsymbol{\omega}$ downstream of a curved shock. Equations (18) and (21) thus complement each other, where Eq. (21) can be viewed as providing vorticity information transverse to the velocity.

Both the velocity and vorticity have a jump discontinuity across a curved shock. The gradient of the velocity also has a discontinuity across the leading and trailing edges of a Prandtl–Meyer expansion. In this circumstance, however, the gradient of the vorticity is continuous. This can be seen from Eq. (18) when the upstream flow is uniform, inviscid, and homentropic. All terms on the right side of Eq. (18) are now zero, and the vorticity is therefore zero throughout the expansion.

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